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# A proof for the informational completeness of the rotated quadrature observables

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**Abstract**

We give a new mathematically rigorous proof for the fact that, when $S$ is a dense subset of $[0, 2\pi)$, the rotated quadrature operators $Q_\theta, \theta \in S$, of a single-mode electromagnetic field constitute an informationally complete set of observables.

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## 1. Introduction

Since the pioneering works of Vogel and Risken [15] and Smithey *et al* [13] the measurement of the rotated quadratures $Q_\theta, \theta \in [0, 2\pi)$, has formed one of the major tools in the quantum state tomography, see, e.g., the compilation [11]. The basic idea behind the state reconstruction is well known: the inverse Radon transform of the quadrature measurement statistics allows one to reconstruct the Wigner function of the state in question and the Wigner function separates states, so that the statistics uniquely determine the state. We do not question the validity of this argument. However, the existing literature, which we are aware of, does not give a full justification that this procedure actually applies to all possible states of a quantum system. Therefore, in this paper, we wish to give a direct proof of the fact that the set of quadrature observables $Q_\theta, \theta \in S, S \subseteq [0, 2\pi)$ is dense and informationally complete, that is, the measurement outcome statistics $p_T^{Q_\theta}$ of these observables uniquely determine the state $T$.

There is a beautiful group theoretical proof of the informational completeness of the observables $Q_\theta, \theta \in [0, 2\pi)$ [6]. This proof builds on a general method of constructing informationally complete sets of observables using the theory of square-integrable representations of unimodular Lie groups. The result in question is then obtained as a special application of this theory to the Weyl–Heisenberg group. Due to the practical importance of the result, we give an alternative direct proof of it. The proof of this fact in section 3 forms the main body of this paper. In the final section we briefly comment on the measurability of the quadrature observables.

## 2. Basic notations and definitions

Let $\mathcal{H}$ be a complex separable Hilbert space, $L(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$ and $\mathcal{T}(\mathcal{H})$ the set of trace class operators. We let $\|\cdot\|_1$ denote the trace norm of $\mathcal{T}(\mathcal{H})$. (The operator norm of $L(\mathcal{H})$ will be denoted by $\|\cdot\|$.) When $\mathcal{H}$ is associated with the quantum system (such as the single-mode electromagnetic field), the states of the system are represented by positive operators $T \in \mathcal{T}(\mathcal{H})$ with unit trace, density operators, and the observables are associated with the normalized positive operator measures defined on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ of the real line[1]. Among them are the conventional von Neumann type of observables, that is, projection-valued measures $P : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$, or, equivalently, selfadjoint operators in $\mathcal{H}$.

The measurement outcome statistics of an observable $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ in a state $T$ is given by the probability measure $X \mapsto \mathrm{tr}[TE(X)] =: p_T^E(X)$.

**Definition 1.** *A set $\mathcal{M}$ of observables $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ is informationally complete, if any two states $S$ and $T$ are equal whenever* $\mathrm{tr}[TE(X)] = \mathrm{tr}[SE(X)]$ *for all $E \in \mathcal{M}$ and $X \in \mathcal{B}(\mathbb{R})$.*

Thus, the informational completeness of a set $\mathcal{M}$ of observables means that the totality of the measurement outcome distributions $p_T^E$, $E \in \mathcal{M}$, determines the state $T$ of the system. Clearly, a set $\mathcal{M}$ of observables is informationally complete if and only if $T = 0$ whenever $T$ is a selfadjoint trace class operator with $\mathrm{tr}[TE(X)] = 0$ for all $E \in \mathcal{M}$ and $X \in \mathcal{B}(\mathbb{R})$. We will use this characterization in our proof.

Fix $\{|n\rangle \mid n \in \mathbb{N}\}$ to be an orthonormal basis of $\mathcal{H}$. (This is identified with the photon number basis in the case where $\mathcal{H}$ is associated with the single-mode electromagnetic field.) Here $\mathbb{N} = \{0, 1, 2, \ldots\}$. We will, without explicit indication, use the coordinate representation, in which $\mathcal{H}$ is represented as $L^2(\mathbb{R})$ via the unitary map $\mathcal{H} \ni |n\rangle \mapsto h_n \in L^2(\mathbb{R})$, where $h_n$ is the $n$th Hermite function,

$$h_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x)\,\mathrm{e}^{-\frac{1}{2}x^2},$$

and $H_n$ is the $n$th Hermite polynomial[2]. Let $a$ and $a^*$ denote the usual raising and lowering operators associated with the above basis of $\mathcal{H}$; they are considered as being defined on their maximal domain

$$D(a) = D(a^*) = \left\{ \varphi \in \mathcal{H} \,\middle|\, \sum_{n=0}^{\infty} n |\langle \varphi | n \rangle|^2 < \infty \right\}.$$

Then define the operators $Q = \frac{1}{\sqrt{2}}\overline{(a^* + a)}$ and $P = \frac{\mathrm{i}}{\sqrt{2}}\overline{(a^* - a)}$, which, in the coordinate representation are the usual multiplication and differentiation operators, respectively: $(Q\psi)(x) = x\psi(x)$, $(P\psi)(x) = -\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}x}(x)$. (Here the bar stands for the closure of an operator, so that, e.g., $Q$ is the unique selfadjoint extension of the essentially selfadjoint symmetric operator $\frac{1}{\sqrt{2}}(a^* + a)$; see [12, chapter IV] or [3, chapter 12] for details concerning the domains of these extensively studied operators.) In the case of the electromagnetic field, $Q$ and $P$ are called the quadrature amplitude operators of the field. In addition, $a = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P)$ and $a^* = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P)$ (see, e.g., [12, p 73]). The Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing $C^\infty$-functions is included in $D(Q) \cap D(P) = D(a)$.

[1] Normalized positive operator measure is a map $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ which is $\sigma$-additive in the weak operator topology, and has the property $E(\mathbb{R}) = I$, that is, for which $X \mapsto \mathrm{tr}[TE(X)]$ is a probability measure for each positive trace one operator $T$.

[2] Hermite polynomials are, of course, given by the following recursion relation: $H_0(x) = 1$, $H_1(x) = 2x$ and $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

For a function $g : \mathbb{R} \to \mathbb{R}$, we let $g^{(k)}$ denote the $k$th derivative of $g$ (with $g^{(0)} = g$), provided it exists. We need the following elementary commutation relations, which hold whenever $\varphi \in \mathcal{S}(\mathbb{R})$, and $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and bounded:

$$(g(Q)a^* - a^*g(Q))\varphi = \tfrac{1}{\sqrt{2}}g^{(1)}(Q)\varphi, \tag{1}$$

$$(g(Q)a - ag(Q))\varphi = -\tfrac{1}{\sqrt{2}}g^{(1)}(Q)\varphi. \tag{2}$$

Let $N$ denote the operator $a^*a$. It is selfadjoint on its natural domain

$$D(N) = \left\{\varphi \in \mathcal{H} \,\middle|\, \sum_{n=0}^{\infty} n^2 |\langle \varphi|n\rangle|^2 < \infty \right\}.$$

Now the phase shifting unitary operators are $\mathrm{e}^{\mathrm{i}\theta N}$, and we can define the *rotated quadrature observables* $Q_\theta$ by

$$Q_\theta = \mathrm{e}^{\mathrm{i}\theta N} Q\, \mathrm{e}^{-\mathrm{i}\theta N}, \qquad \theta \in [0, 2\pi).$$

The spectral measure of $Q_\theta$ will be denoted by $P^{Q_\theta} : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$.

## 3. The proof

We need the so-called *Dawson's integral*

$$\mathrm{daw}(x) = \mathrm{e}^{-x^2} \int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t$$

(see, e.g., [1, pp 298, 299] or [14, chapter 42]). The following lemma lists those properties of Dawson's integral that we are going to use. Since the Dawson's integral has been studied extensively, they are probably well known. However, as we were unable to find these results directly stated in the literature, we give a proof in the appendix; a reader familiar with the results may skip that proof.

**Lemma 1.**

*(a)* daw : $\mathbb{R} \to \mathbb{R}$ *is a* $C^\infty$*-function, and*

$$\lim_{x\to\pm\infty} \mathrm{daw}^{(k)}(x) = 0 \qquad \textit{for all} \quad k \in \mathbb{N}.$$

*(b)*

$$\mathrm{daw}^{(1)}(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n(2n)!} H_{2n}(x) \qquad \textit{for all} \quad x \in \mathbb{R},$$

*where* $H_n$ *is the nth Hermite polynomial.*

**Lemma 2.** *There exists a* $C^\infty$*-function* $f : \mathbb{R} \to \mathbb{R}$*, such that*

*(i) each derivative* $f^{(n)}$ *of* $f$*,* $n = 0, 1, 2, \ldots$ *is a bounded function;*
*(ii)* $\langle n|f(Q)|n\rangle = \delta_{n0}$*, for all* $n \in \mathbb{N}$*, where* $\delta$ *is the Kronecker delta.*

**Proof.** We put $f = 2\,\mathrm{daw}^{(1)}$. According to lemma 1, $f$ is a $C^\infty$-function and (i) holds. Lemma 1 (b) gives the expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n(2n)!} H_{2n}(x), \qquad x \in \mathbb{R}. \tag{3}$$

Since the series in (3) converges pointwise, we get

$$f(x)h_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n (2n)!} \sqrt{2^{2n}(2n)!}\, h_{2n}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\sqrt{(2n)!}} h_{2n}(x)$$

for each $x \in \mathbb{R}$. In addition, $\sum_{n=0}^{\infty} (n!)^2/(2n)! < \infty$, so the series also converges in $L^2$-norm. Noting also that $H_n^2 h_0 \in L^2(\mathbb{R})$, we can justify the following computation:

$$\begin{aligned}
\langle h_n | f(Q) h_n\rangle &= \frac{1}{2^n n!} \langle H_n^2 h_0 | f h_0\rangle = \frac{1}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{2^k (2k)!} \big\langle H_n^2 h_0 \big| \sqrt{2^{2k}(2k)!}\, h_{2k}\big\rangle \\
&= \frac{1}{2^n n!} \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k (2k)!} \big\langle H_n^2 h_0 \big| H_{2k} h_0\big\rangle \\
&= \frac{1}{2^n n!} \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k (2k)! \sqrt{\pi}} \int_{\mathbb{R}} H_{2k}(x) (H_n(x))^2 \,\mathrm{e}^{-x^2}\, \mathrm{d}x \\
&= \frac{1}{2^n n!} \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k (2k)! \sqrt{\pi}} \frac{2^{k+n} \sqrt{\pi} (2k)! (n!)^2}{(n-k)! (k!)^2} = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)! k!} \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \lim_{y \to 1} (1-y)^n = \delta_{n0}.
\end{aligned}$$

Here the finite sum after the third equality is obtained by noting that $h_{2k}$ is orthogonal to $H_n^2 h_0$ whenever $k > n$, which is due to the fact that the latter function is a linear combination of Hermite functions $h_0, \ldots, h_{2n}$. The fifth equality follows from formula 7.375(2) of [7, p 838]. □

Now we choose a function $f$ satisfying the conditions of lemma 2. The function $f$ will remain fixed throughout the rest of the paper.

**Lemma 3.** $\langle n+k | f^{(k)}(Q) | n\rangle = (-1)^k \sqrt{2^k k!}\, \delta_{0n}$ *for all* $k, n \in \mathbb{N}$.

**Proof.** We proceed by induction with respect to $k$; the initial step is provided by condition (ii) of lemma 2. The induction assumption is that for some $k \in \mathbb{N}$, the equality

$$\langle n+k | f^{(k)}(Q) | n\rangle = (-1)^k \sqrt{2^k k!}\, \delta_{0n}$$

holds for all $n \in \mathbb{N}$. We must show that

$$\langle n+k+1 | f^{(k+1)}(Q) | n\rangle = (-1)^{k+1} \sqrt{2^{k+1}(k+1)!}\, \delta_{0n}, \qquad n \in \mathbb{N}.$$

But by using (1), we get

$$\begin{aligned}
\langle n+k+1 | f^{(k+1)}(Q) | n\rangle &= \sqrt{2} \langle n+k+1 | f^{(k)}(Q) a^* | n\rangle - \sqrt{2} \langle n+k+1 | a^* f^{(k)}(Q) | n\rangle \\
&= \sqrt{2(n+1)} \langle (n+1)+k | f^{(k)}(Q) | n+1\rangle - \sqrt{2(n+k+1)} \langle n+k | f^{(k)}(Q) | n\rangle \\
&= -\sqrt{2(n+k+1)} (-1)^k \sqrt{2^k k!}\, \delta_{0n} = (-1)^{k+1} \sqrt{2^{k+1}(k+1)!}\, \delta_{0n}
\end{aligned}$$

for all $n \in \mathbb{N}$ by the induction assumption. (Note, in particular, that the first term in the expression following the second equality is indeed zero by the induction assumption, because $n+1 > 0$.) □

**Lemma 4.**

$$\langle n+k | f^{(k+2l)}(Q) | n\rangle = 2^l (-1)^k \sqrt{2^k l! (l+k)!}\, \delta_{ln}, \qquad k, l, n \in \mathbb{N}, \quad n \geqslant l.$$

**Proof.** Now we use induction with respect to $l$, so the initial step $l = 0$ is given by lemma 3. The induction assumption is that

$$\langle n+k|f^{(k+2l)}(Q)|n\rangle = 2^l(-1)^k\sqrt{2^k l!(l+k)!}\delta_{ln}$$

holds for some $l \in \mathbb{N}$, *all* $k \in \mathbb{N}$, and *all* $n \geqslant l$. We have to show that this also holds when $l$ is replaced by $l+1$. Accordingly, let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $n \geqslant l+1$. Using the commutation relation (2) and the induction assumption, we get

$$\begin{aligned}\langle n+k|f^{(k+2(l+1))}(Q)|n\rangle &= -\sqrt{2}\langle n+k|f^{(k+2l+1)}(Q)a|n\rangle + \sqrt{2}\langle n+k|af^{(k+2l+1)}(Q)|n\rangle \\ &= -\sqrt{2n}\langle (n-1)+(k+1)|f^{((k+1)+2l)}(Q)|n-1\rangle \\ &\quad + \sqrt{2(n+k+1)}\langle n+(k+1)|f^{((k+1)+2l)}(Q)|n\rangle \\ &= -\sqrt{2n}2^l(-1)^{k+1}\sqrt{2^{k+1}l!(l+k+1)!}\delta_{l,n-1} \\ &= (-1)^k\sqrt{2(l+1)}2^l\sqrt{2}\sqrt{2^k l!(l+1+k)!}\delta_{l+1,n} \\ &= 2^{l+1}(-1)^k\sqrt{2^k(l+1)!(l+1+k)!}\delta_{l+1,n}.\end{aligned}$$

Here the induction assumption was applied to the first term following the second equality with $n$ and $k$ replaced by $n-1$ and $k+1$, and to the second term with $n$ and $k$ replaced by $n$ and $k+1$. Here it is essential to note that $n \geqslant l+1 > l$, which makes the second term zero. □

In order to construct the proof for the informational completeness of the quadratures, we need some additional tools. First define, for each fixed $k \in \mathbb{N}$ and $X \in \mathcal{B}(\mathbb{R})$,

$$V^k(X) := \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta} P^{Q_\theta}(X)\,\mathrm{d}\theta \in L(\mathcal{H}),$$

where the integral is to be understood in the $\sigma$-weak operator topology. Indeed, for each trace class operator $T$, the map $\theta \mapsto \mathrm{tr}[TP^{Q_\theta}(X)] = \mathrm{tr}[T\,\mathrm{e}^{\mathrm{i}\theta N}P^Q(X)\,\mathrm{e}^{-\mathrm{i}\theta N}]$ is continuous[3], and $|\mathrm{tr}[TE^{Q_\theta}(X)]| \leqslant \|T\|_1\|P^{Q_\theta}(X)\| \leqslant \|T\|_1$, so the integral exists in the $\sigma$-weak sense, and represents a bounded operator, with $\|V^k(X)\| \leqslant 2\pi$.

Next, note that for each $k \in \mathbb{N}$, the map $X \mapsto V^k(X)$ is an operator measure, that is, $\sigma$-additive in the weak operator topology. In fact, if $(X_n)_{n\in\mathbb{N}}$ is a sequence of mutually disjoint sets in $\mathcal{B}(\mathbb{R})$, then for any $l \in \mathbb{N}$,

$$\left|\sum_{n=0}^{l} \mathrm{e}^{-\mathrm{i}k\theta}\langle\varphi|P^{Q_\theta}(X_n)\varphi\rangle\right| \leqslant \sum_{n=0}^{l}\langle\varphi|P^{Q_\theta}(X_n)\varphi\rangle \leqslant \langle\varphi|P^{Q_\theta}\big(\cup_{n=0}^\infty X_n\big)\varphi\rangle \leqslant \|\varphi\|^2,$$

so the dominated convergence theorem can be applied to get

$$\begin{aligned}\sum_{n=0}^{\infty}\langle\varphi|V^k(X_n)\varphi\rangle &= \sum_{n=0}^{\infty}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}k\theta}\langle\varphi|P^{Q_\theta}(X_n)\varphi\rangle\,\mathrm{d}\theta = \int_0^{2\pi}\mathrm{e}^{-\mathrm{i}k\theta}\langle\varphi|P^{Q_\theta}(\cup_{n=0}^\infty X_n)\varphi\rangle\,\mathrm{d}\theta \\ &= \big\langle\varphi\big|V^k\big(\cup_{n=0}^\infty X_n\big)\varphi\big\rangle.\end{aligned}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be any bounded Borel function. Then the operator integral $V^k[g] = \int g\,\mathrm{d}V^k$ can be defined in the $\sigma$-weak sense, as a bounded operator. This follows from the fact that $g$ is bounded and

$$\big|V_T^k\big|(\mathbb{R}) \leqslant 4\sup_{X\in\mathcal{B}(\mathbb{R})}|\mathrm{tr}[TV^k(X)]| \leqslant 8\pi\|T\|_1$$

[3] This can easily be seen by, e.g., considering a positive trace class operator $T$, using its spectral resolution and applying the strong continuity of the map $\theta \mapsto \mathrm{e}^{\mathrm{i}\theta N}$.

for any trace class operator $T$, with $\left|V_T^k\right|$ denoting the total variation of the complex measure $V_T^k = \mathrm{tr}[TV^k(\cdot)]$ (see, e.g., [5, p 97]). (The map $X \mapsto \mathrm{tr}[TV^k(X)]$ is a complex measure, because the weak and $\sigma$-weak operator topologies coincide in a norm-bounded set.)

**Lemma 5.** *For any bounded function* $g : \mathbb{R} \to \mathbb{R}$, *and a trace class operator T, we have*

$$\mathrm{tr}[TV^k[g]] = 2\pi \sum_{n=0}^{\infty} \langle n|T|n+k\rangle\langle n+k|g(Q)|n\rangle.$$

**Proof.** First note that $\langle m|V^k[g]|n\rangle = \mathrm{tr}[|n\rangle\langle m|V^k[g]] = \int g \, \mathrm{d}V^k_{|n\rangle\langle m|}$ by definition. On the other hand, for each $X \in \mathcal{B}(\mathbb{R})$, we get

$$V^k_{|n\rangle\langle m|}(X) = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta}\langle m|P^{Q_\theta}(X)|n\rangle \, \mathrm{d}\theta = \langle m|P^Q(X)|n\rangle \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta}\, \mathrm{e}^{\mathrm{i}\theta(m-n)} \, \mathrm{d}\theta.$$

This equals $2\pi\langle n+k|P^Q(X)|n\rangle$ if $m = n+k$ and zero otherwise. Hence,

$$\langle n+k|V^k[g]|n\rangle = 2\pi \int g \, \mathrm{d}P^Q_{|n\rangle\langle n+k|} = 2\pi\langle n+k|g(Q)|n\rangle,$$

and $\langle m|V^k[g]|n\rangle = 0$ whenever $m \neq n+k$. Thus,

$$\begin{aligned}\mathrm{tr}[TV^k[g]] &= \sum_{n=0}^{\infty}\langle n|TV^k[g]|n\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle n|T|m\rangle\langle m|V^k[g]|n\rangle \\ &= 2\pi\sum_{n=0}^{\infty}\langle n|T|n+k\rangle\langle n+k|g(Q)|n\rangle.\end{aligned}$$

□

Now we are ready to prove the actual result of the paper.

**Theorem 1.** *Let S be a dense subset of* $[0, 2\pi)$. *The set* $\{P^{Q_\theta} \mid \theta \in S\}$ *of observables is informationally complete.*

**Proof.** Let $T \in L(\mathcal{H})$ be a selfadjoint trace class operator, such that $\mathrm{tr}[TP^{Q_\theta}(X)] = 0$ for all $X \in \mathcal{B}(\mathbb{R})$ and $\theta \in S$. Since $\theta \mapsto \mathrm{tr}[TP^{Q_\theta}(X)]$ is continuous and $S$ is dense it follows that $\mathrm{tr}[TP^{Q_\theta}(X)] = 0$ for all $X \in \mathcal{B}(\mathbb{R})$ and $\theta \in [0, 2\pi)$.

Let $k \in \mathbb{N}$ be fixed. By the definition of the operator measure $V^k$, the assumption implies that

$$V_T^k(X) = \mathrm{tr}[TV^k(X)] = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta}\mathrm{tr}[TP^{Q_\theta}(X)] \, \mathrm{d}\theta = 0$$

for all $X \in \mathcal{B}(\mathbb{R})$. From the definition of the $\sigma$-weak integral $V^k[g] = \int g \, \mathrm{d}V^k$, it follows that $\mathrm{tr}[TV^k[g]] = \int g \, \mathrm{d}V_T^k = 0$ for any bounded Borel function $g : \mathbb{R} \to \mathbb{R}$. Hence, by using lemma 5, we get

$$\sum_{n=0}^{\infty}\langle n|T|n+k\rangle\langle n+k|g(Q)|n\rangle = 0 \tag{4}$$

for any bounded Borel function $g : \mathbb{R} \to \mathbb{R}$. We show by induction with respect to $n$ that $\langle n|T|n+k\rangle = 0$ for all $n \in \mathbb{N}$. First, put $g = f^{(k)}$ in (4) (recall that $f$ was a function satisfying the assumptions of lemma 2). By lemma 4, this gives $\langle 0|T|k\rangle = 0$, which proves the initial step. The induction assumption is that for some $m \in \mathbb{N}$, $\langle n|T|n+k\rangle = 0$ for all $n \in \mathbb{N}$, $n \leqslant m$. We have to show that this implies $\langle m+1|T|(m+1)+k\rangle = 0$. By the induction assumption and (4), we have

$$\sum_{n=m+1}^{\infty} \langle n|T|n+k\rangle\langle n+k|f^{(k+2(m+1))}(Q)|n\rangle = 0,$$

where we have simply put $g = f^{(k+2(m+1))}$, which is again a bounded Borel function. But, according to lemma 4,

$$\langle n+k|f^{(k+2(m+1))}(Q)|n\rangle = 0, \qquad n > m+1,$$

and

$$\langle (m+1)+k|f^{(k+2(m+1))}(Q)|m+1\rangle \neq 0,$$

so that necessarily $\langle m+1|T|(m+1)+k\rangle = 0$. This completes the induction proof.

We have thus established that $\langle l|T|l+k\rangle = 0$ for all $l, k \in \mathbb{N}$. Since $T$ is selfadjoint, this implies that $T = 0$ and the proof is complete. □

Note that the set $S$ in the previous theorem can be chosen to be countable (e.g., $S = [0, 2\pi) \cap \mathbb{Q}$). Hence, in principle, it suffices to measure a *sequence* of quadrature observables in order to determine the state of the system.

Though obvious, it may be worth noting that the quadrature observables $Q_\theta$ are not informationally complete in the sense of statistical expectation, that is, the numbers $\mathrm{tr}[TQ_\theta]$, $\theta \in [0, 2\pi)$, do not, in general, determine the state $T$; for instance, the number states $|n\rangle$ are indistinguishable by the expectations, $\langle n|Q_\theta|n\rangle = 0$ for all $n$ and $\theta$.

## 4. Concluding remarks

According to the result in the preceding section, the quadrature observables $Q_\theta, \theta \in S$ ($S \subseteq [0, 2\pi)$ is dense) constitute an informationally complete set of observables, i.e. the measurement statistics $p_T^{Q_\theta}, \theta \in S$, determine uniquely the state $T$ of the quantum system. The question of experimental implementation of these observables is thus of utmost importance.

The balanced homodyne detection with a strong auxiliary field is a well-developed method of experimental quantum physics, and this method is known to yield the measurement statistics of the quadrature observable $Q_\theta$, depending on the phase $\mathrm{e}^{\mathrm{i}\theta}$ of the (one-mode) auxiliary field. The heuristic physical argument behind this method is equally well known, see, e.g., [8, 10], the detailed mathematical justification being, however, more involved.

If $|z\rangle$, $z = r\mathrm{e}^{\mathrm{i}\theta}$, is the coherent state of the (one-mode) auxiliary field, then the actually measured observable in the balanced homodyne detection is given by a semispectral measure $E^z$, whose first moment operator $E^z[1]$ is an extension of the restriction of the quadrature operator $Q_\theta$ on the domain $D(a)$ of the signal mode operator $a$, and whose noise operator $E^z[2] - E^z[1]^2$ equals with the operator $\frac{1}{2}r^{-2}N$ (where $N = a^*a$). Clearly, these results suggest that the high-amplitude limit of $E^z$ is the spectral measure $P^{Q_\theta}$ of $Q_\theta$, notably since the spectral measures are known to be exactly those semispectral measures whose noise operators equal to zero. There is, indeed, a rigorous quantum-mechanical proof of the fact that in the high-amplitude limit the observable $E^z$ tends to the spectral measure of $Q_\theta$, though the actual meaning of this limit requires more caution [9].

The balanced homodyne detection scheme can thus be used to collect the statistics of the quadrature observables. According to the result proved in this paper these statistics determine the state of the system.

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## Appendix. The proof of lemma 1

To begin the proof of the lemma, we first note that the Dawson's integral is clearly a $C^\infty$-function. We prove (a) by using the expansion

$$\mathrm{daw}(x) = \frac{1}{2x}\sum_{j=0}^{n-1}\frac{(2j-1)!!}{(2x^2)^j} + \frac{(2n-1)!!}{2^n}R_n(x),$$

where

$$R_n(x) = \frac{\mathrm{e}^{-x^2}}{x^{2n-1}}\sum_{j=0}^{\infty}\frac{x^{2j}}{j!(2j-2n+1)}$$

and $n = 1, 2, 3, \ldots$ (see [14, equation 42:6:5, p 407]). Since each derivative is either even or odd, it suffices to consider the limit $x \to \infty$. Clearly, any derivative of the first part tends to zero at this limit, for any choice of $n$. As for $R_n$, it is easy to see that for any given $k$, we get $\lim_{x\to\infty} R_n^{(k)}(x) = 0$ for sufficiently large $n$. Indeed, let

$$S_n(y) = \sum_{j=0}^{\infty}\frac{y^j}{j!(2j-2n+1)},$$

so that $R_n(x) = \frac{\mathrm{e}^{-x^2}}{x^{2n-1}}S_n(x^2)$. Since the power series $S_n(y)$ is clearly convergent for all $y \in \mathbb{R}$, it can be differentiated $k$ times (for any $k \in \mathbb{N}$) to get

$$S_n^{(k)}(y) = \sum_{j=k}^{\infty}\frac{y^{j-k}}{(j-k)!(2j-2n+1)}.$$

From this we see that $\left|S_n^{(k)}(y)\right| \leqslant \mathrm{e}^y$ for any $y > 0$ and $n \in \mathbb{N}$. Now for a fixed $k \in \mathbb{N}$, put, e.g., $n = k+1$. Since $R_n^{(k)}(x)$ is clearly a finite sum of terms of the form $A_{l,l'}\frac{\mathrm{e}^{-x^2}}{x^{2n-1-l}}S_n^{(l')}(x^2)$, with $-k \leqslant l \leqslant k$, $0 \leqslant l' \leqslant k$, and $A_{l,l'}$ a constant, it follows that $\lim_{x\to\infty} R_n^{(k)}(x) = 0$. The proof of (a) is complete.

To prove (b), consider the series

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n n!}{2^n(2n)!}H_{2n}(x). \tag{A.1}$$

We first note that the well-known relation $\frac{\mathrm{d}}{\mathrm{d}x}H_l(x) = 2lH_{l-1}(x)$, $l = 1, 2, 3, \ldots$, implies

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}H_{2n}(x) = 2n(2n-1)\cdots(2n-m+1)2^m H_{2n-m}(x), \qquad m \leqslant 2n. \tag{A.2}$$

Using the estimate $|H_n(x)| \leqslant e^{\frac{1}{2}x^2} K 2^{\frac{1}{2}n}\sqrt{n!}$, where $K > 0$ is a constant ([1, 22.14.17, p 787]), we get

$$\left|\frac{(-1)^n n!}{2^n(2n)!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}H_{2n}(x)\right| \leqslant 2^{\frac{m}{2}} K \frac{n!2n(2n-1)\cdots(2n-m+1)\sqrt{(2n-m)!}}{(2n)!}\,e^{\frac{1}{2}x^2}$$

$$\leqslant 2^{\frac{m}{2}} K \frac{(2n)^m n!}{\sqrt{(2n)!}}\,e^{\frac{1}{2}x^2}$$

for all $n, m \in \mathbb{N}, m \leqslant 2n$. Now $\sum_{n=0}^{\infty} \frac{(2n)^m n!}{\sqrt{(2n)!}} < \infty$ for any $m \in \mathbb{N}$ by the ratio test, which shows that the series obtained by differentiating (A.1) $m$ times term by term converges uniformly in bounded intervals. Consequently, that series represents the $m$th derivative of the original series (A.1), the latter converging to some $C^\infty$-function $F : \mathbb{R} \to \mathbb{R}$ uniformly in bounded intervals.

By using again the formula $\frac{\mathrm{d}}{\mathrm{d}x}H_l(x) = 2lH_{l-1}(x)$, we get

$$2F^{(2m)}(x) = 2^{2m}\sum_{n=m}^{\infty}\frac{n!(-1)^n H_{2(n-m)}(x)}{2^n(2(n-m))!}; \tag{A.3}$$

$$2F^{(2m+1)}(x) = 2^{2m+1}\sum_{n=m+1}^{\infty}\frac{n!(-1)^n H_{2(n-m)-1}(x)}{2^n(2(n-m)-1)!}. \tag{A.4}$$

In order to calculate the MacLaurin series of $F$, we need the expansion

$$(1-x)^{-(m+1)} = \sum_{n=0}^{\infty}\binom{m+n}{n}x^n, \qquad -1 < x < 1, \tag{A.5}$$

which can be obtained from the binomial series (see 3.6.9 in [1, p 15]).

Now, using (A.3), the identity $H_{2(n-m)}(0) = (-1)^{n-m}(2(n-m))!/(n-m)!, n \geqslant m$ (22.4.8 in [1, p 777]), and (A.5) with $x = \frac{1}{2}$, we get

$$2F^{(2m)}(0) = 2^{2m}(-1)^m\sum_{n=m}^{\infty}\frac{n!}{2^n(n-m)!} = 2^m(-1)^m\sum_{n=0}^{\infty}\frac{(n+m)!}{2^n n!}$$

$$= 2^m(-1)^m m!\sum_{n=0}^{\infty}\binom{n+m}{n}\frac{1}{2^n} = 2^{2m+1}(-1)^m m!.$$

Since $F^{(2m+1)}(0) = 0$ by (A.4) and the fact that $H_l(0) = 0$ for odd $l$, we get the following MacLaurin series for $F$:

$$\sum_{m=0}^{\infty}\frac{(-1)^m m!}{(2m)!}(2x)^{2m}, \qquad x \in \mathbb{R}. \tag{A.6}$$

This is exactly the MacLaurin series for the first derivative of the Dawson's integral, since

$$\mathrm{daw}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m m! 2^{2m}}{(2m+1)!}x^{2m+1}$$

(see, e.g., [14, p 406]).

In order to prove that series (A.6) actually converges to $F$ (pointwise for all $x \in \mathbb{R}$), we have to show that for each $x \in \mathbb{R}$, the corresponding remainder $F^{(k)}(\xi_k)x^k/k!$, where

$\xi_k \in [-|x|, |x|]$, goes to zero as $k \to \infty$. By applying the estimate $|H_l(y)| \leqslant K e^{\frac{1}{2}y^2} 2^{l/2}\sqrt{l!}$ to (A.3) and (A.4), we get

$$2|F^{(2m)}(y)| \leqslant e^{\frac{1}{2}y^2} 2^m K \sum_{n=0}^{\infty} \frac{(n+m)!}{\sqrt{(2n)!}} = K e^{\frac{1}{2}y^2} 2^m m! \sum_{n=0}^{\infty} \binom{n+m}{n} \frac{n!}{\sqrt{(2n)!}};$$

$$\begin{aligned} 2|F^{(2m+1)}(y)| &\leqslant e^{\frac{1}{2}y^2} \sqrt{2} 2^m K \sum_{n=0}^{\infty} \frac{(n+m+1)!}{\sqrt{(2n+1)!}} \\ &= \sqrt{2} K e^{\frac{1}{2}y^2} 2^m (m+1)! \sum_{n=0}^{\infty} \binom{n+m+1}{n} \frac{n!}{\sqrt{(2n+1)!}} \end{aligned}$$

for all $y \in \mathbb{R}$. It is easy to see by induction that $n!/\sqrt{(2n)!} \leqslant n/2^{n-1}$ for all $n \geqslant 1$; using this, as well as the relation $(m+1)2^{m+1} = \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{n}{2^n}$ which is obtained by differentiating (A.5) and putting $x = \frac{1}{2}$, we conclude that $2|F^{(2m)}(y)| \leqslant 4K e^{\frac{1}{2}y^2} 2^{2m}(m+1)!$, and $2|F^{(2m+1)}(y)| \leqslant 4\sqrt{2} K e^{\frac{1}{2}y^2} 2^{2m+1}(m+2)!$ for all $y \in \mathbb{R}$. Consequently,

$$\frac{|F^{(2m)}(\xi)x^{2m}|}{(2m)!} \leqslant 2K e^{\frac{1}{2}x^2} \frac{(2|x|)^{2m}(m+1)!}{(2m)!},$$

$$\frac{|F^{(2m+1)}(\xi)x^{2m+1}|}{(2m+1)!} \leqslant 2\sqrt{2} K e^{\frac{1}{2}x^2} \frac{(2|x|)^{2m+1}(m+2)!}{(2m+1)!},$$

whenever $x \in \mathbb{R}$, $\xi \in [-|x|, |x|]$ and $m \in \mathbb{N}$. It is easy to see that for each fixed $x \in \mathbb{R}$, the right-hand sides of the inequalities tend to zero as $m \to \infty$. Hence, we have shown that

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^m m!}{(2m)!} (2x)^{2m} = \mathrm{daw}^{(1)}(x), \qquad x \in \mathbb{R}.$$

This completes the proof of lemma 1.

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